

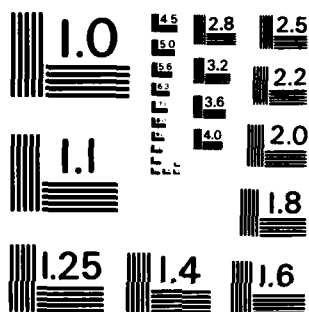
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An Algorithm for Least Squares Projections  
onto the Intersection of Shifted, Convex Cones

by

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ABSTRACT

A commonly occurring problem is that of minimizing least squares expressions subject to restrictions on the solution. Dykstra (1983) has given a simple algorithm for solving these types of problems when the constraint region can be expressed as a finite intersection of closed, convex cones. Here it is shown that this algorithm must still work correctly even when each cone is allowed to be arbitrarily translated (as long as the intersection is non-empty). This allows the algorithm to be applied to a much larger collection of problems than previously indicated.

Key words and phrases: Least squares projections, regression, cones constraints, translations, minimizations, dual cones, iterations.

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An Algorithm for Least Squares Projections  
onto the Intersection of Shifted Convex Cones

Richard L. Dykstra

1. Introduction.

The problem of obtaining least squares projections subject to various constraints is a frequently occurring problem in many areas. For example, the area of isotonic regression usually is concerned with obtaining least squares vectors which must satisfy certain partial order restrictions (see Barlow, Bartholomew, Bremner and Brunk (1972) and Robertson and Wright (1981)). Another example concerns finding the closest (least squares) convex (concave) function through a set of points in the plane (see Hildreth (1954) and Wu (1982)).

Many times, the constraint region can be written as a finite intersection of simpler constraint regions. This raises the possibility of using iterative schemes based upon the projections onto the simpler regions for solving the overall problem.

John Von Neumann (1950) has shown that if the constraint region is an intersection of two subspaces, cyclic iterative projections onto the individual subspaces must converge to the desired projection. Norbert Wiener (1955) independently proved a version of this theorem in a slightly different setting.

Dykstra and Robertson (1982) have developed an iterative procedure for finding projections of rectangular arrays onto the class of arrays with nondecreasing rows and columns based only upon one-dimensional smoothings. Later, Dykstra (1983) extended this approach to the general framework of projections onto the intersection of closed convex cones. This procedure is based upon finding projections onto the individual cones, and reduces to Von Neumann's and Weiner's method when the cones are also subspaces.

It is the purpose of this paper to show that Dykstra's (1983) algorithm can be extended to work for projections onto a finite non-empty intersection of shifted (translated) closed, convex cones. In particular, this means that least squares projections under general linear constraints of the form  $\sum_{i=1}^n a_i x_i \leq (=) b$  can be handled by Dykstra's algorithm even when  $b \neq 0$ . This clearly follows by writing  $\{x: \sum_{i=1}^n a_i x_i \leq b\}$  as

$$\{x: \sum_{i=1}^n a_i x_i \leq 0\} - (-b/a_1, 0, \dots, 0).$$

Constraints of the form

$$y_i \leq x_i \leq z_i, \quad i = 1, \dots, n$$

where  $y$  and  $z$  are fixed vectors fall into the translated cone framework by writing

$$\{x: y_i \leq x_i \leq z_i \quad \forall_i\}$$

as

$$(\{x: x_i \geq 0\} - (-y_1, -y_2, \dots, -y_n)) \cap (\{x: x_i \leq 0\} - (-z_1, -z_2, \dots, -z_n)).$$

Other examples of cone constraints given in Dykstra (1983) can be generalized to translated cone constraints.

## 2. Notation and Setting.

We denote  $n$ -dimensional real coordinate space by  $R^n$ , and let  $g$  and  $w(w_i > 0)$  be fixed points in  $R^n$ . The inner product of  $x$  and  $y$  (with respect to  $w$ ) is given by

$$(x, y) = \sum_{i=1}^n x_i y_i w_i.$$

The corresponding inner product norm of  $x$  is defined as

$$(2.1) \quad \|x\| = (x, x)^{\frac{1}{2}} = \left( \sum_{i=1}^n x_i^2 w_i \right)^{\frac{1}{2}}.$$

A closed (in the metric) subset  $K$  of  $R^n$  is a closed convex cone if  $x, y \in K$ ;  $a, b \geq 0$  implies  $ax + by \in K$ . The dual cone of a closed, convex cone  $K$  is defined as

$$(2.2) \quad K^* = \{y \in R^n; (y, x) = \sum_{i=1}^n y_i x_i w_i \leq 0 \quad \forall x \in K\}.$$

Of course  $K^*$  is also a closed, convex cone with the property that  $K^{**} = K$ .

A commonly occurring problem is to find the  $x$  which will

$$(2.3) \quad \underset{x \in C}{\text{Minimize}} \quad \|g - x\|$$

where  $C$  is a closed, convex set ( $x, y \in C$  implies  $\alpha x + (1-\alpha)y \in C$   $\forall \alpha \in [0,1]$ ). A vector  $g^* \in C$  achieves the minimal value in (2.3) iff

$$(2.4) \quad (g - g^*, g^* - f) \geq 0 \quad \text{for all } f \in C.$$

If  $C$  is actually a closed, convex cone, we may replace (2.4) by

$$(2.5) \quad \begin{array}{ll} \text{i)} & (g - g^*, g^*) = 0, \quad \text{and} \\ \text{ii)} & (g - g^*, f) \leq 0 \quad \text{for all } f \in C. \end{array}$$

(See Theorem 7.8 of Barlow et al. (1972).)

Note that (2.5) implies that if  $C$  is a closed, convex cone and  $g^*$  solves (2.3), then  $g - g^* \in C^*$ .

### 3. The Algorithm.

We wish to consider problems of the form

$$(3.1) \quad \underset{x \in \bigcap_{i=1}^F (K_i - b_i)}{\text{Minimize}} \quad \|g - x\|$$



where  $K_i$  is a closed, convex cone in  $R^n$ ,  $b_i \in R^n$ ,

$K_i - b_i = \{x - b_i; x \in K_i\}$ , and  $\bigcap_{i=1}^r (K_i - b_i) \neq \emptyset$ . We assume that we can find the vector in  $K_i - b_i$  which will

$$(3.2) \quad \begin{array}{l} \text{Minimize } \|f - x\| \\ x \in K_i - b_i \end{array}$$

for any  $f$  and any  $i$ , and wish to use this ability to solve (3.1).

We note that if  $P(f|C)$  denotes the projection of  $f \in R^n$  onto the closed, convex set  $C$ , then for a closed, convex cone  $K$

$$(3.3) \quad P(f|K_i - b_i) = P(f + b_i|K_i) - b_i.$$

Thus

$$(3.4) \quad f - P(f|K_i - b_i) = (f + b_i) - P(f + b_i|K_i) \in K_i^* \quad \text{for all } f$$

by (2.5) ii).

We shall make extensive use of (3.4).

Our proposed algorithm is identical to that given in Dykstra (1983) except that we allow projections onto shifted, closed, convex cones.

Our scheme can be succinctly stated with the aid of the following notation:

- i) For any positive integer  $n$ , we define  $n(\bmod r) = i$  if  $n = kr + i$  for integers  $k$  and  $i$  where  $1 \leq i \leq r$ .
- ii) Initially, set  $n = 0$ ,  $g_0 = g$ , and  $I_i = 0 \in R^n$ ,  $i = 1, \dots, r$ .

The iterative procedure is to

- 1) Set  $g_{n+1} = P(g_n - I_{n(\text{mod } r)} | K_{n(\text{mod } r)} - b_{n(\text{mod } r)})$ , and  
 then update  $I_{n(\text{mod } r)}$  by resetting it equal to  
 (3.5)  $g_{n+1} - (g_n - I_{n(\text{mod } r)})$ .  
 2) Replace  $n$  by  $n+1$  and go to 1).

We refer the reader to Dykstra (1983) for further elaboration on the algorithm and its uses. This procedure requires only the ability to find projections onto the  $K_i$  (see (3.3)). These individual projections are often easy to program and quick to execute, and hence can be combined to solve rather difficult optimization problems.

In particular, the algorithm applies to quadratic programming problem with a finite number of constraints of the form  $\sum_{i=1}^n a_i x_i \leq b$ .

#### 4. Proof of the Algorithm.

To simplify the proof for  $n \geq 1$ , we will write  $g_n$  in (3.5) as  $g_{k,i}$  when  $n = (k-1)r + i$ ,  $1 \leq i \leq r$ . (In other words,  $g_{k,i}$  is the projection onto the  $i^{\text{th}}$  shifted cone during the  $k^{\text{th}}$  cycle.)

In similar fashion we depict

$$(4.1) \quad I_{k,i} = \begin{cases} g_{k,i} - (g_{k,i-1} - I_{k-1,i}), & \text{if } 2 \leq i \leq r \\ g_{k,1} - (g_{k-1,r} - I_{k-1,1}), & \text{if } i = 1. \end{cases}$$

We will also have need of the following lemma, a proof of which is found in Dykstra (1983).

Lemma 4.1. Suppose a sequence of nonnegative real numbers  $\{a_n\}_{n=1}^{\infty}$  is such that  $\sum_{n=1}^{\infty} a_n^2 < \infty$ . Then there exists a subsequence  $\{a_{n_j}\}_{j=1}^{\infty}$  such that

$$\sum_{m=1}^{n_j} a_m a_{n_j} \rightarrow 0 \text{ as } j \rightarrow \infty.$$

We now establish the fundamental result of the paper.

Theorem 4.1. The vectors  $g_n$  defined in (3.5) converge to the true solution, say  $g^*$ , of the problem defined in (3.1) as  $n \rightarrow \infty$ .

Proof. Since  $\bigcap_{i=1}^r (K_i - b_i) \neq \emptyset$ , we may assume WLOG that  $0 \in \bigcap_{i=1}^r (K_i - b_i)$ . Then  $b_i \in K_i$  and the true solution  $g^*$  exists uniquely. Note from (4.1) that

$$g_{n,i-1} - g_{n,i} = I_{n-1,i} - I_{n,i}, \quad i = 2, \dots, r, \text{ and} \quad (4.2)$$

$$g_{n-1,r} - g_{n,1} = I_{n-1,1} - I_{n,1}.$$

Thus, in general  $(I_{0,i} = 0)$ , for  $i \geq 2$

$$\begin{aligned} \|g_{n,i-1} - g^*\|^2 &= \|(g_{n,i} - g^*) + (I_{n-1,i} - I_{n,i})\|^2 \\ (4.3) \quad &= \|g_{n,i} - g^*\|^2 + \|I_{n-1,i} - I_{n,i}\|^2 \\ &\quad + 2(g_{n,i} + b_i, I_{n-1,i} - I_{n,i}) - 2(g^* + b_i, I_{n-1,i} - I_{n,i}). \end{aligned}$$

Note that  $(g_{n,i} + b_i, I_{n,i}) = 0$  (by (2.5)).

Moreover, since  $g_{n,i} + b_i \in K_i$  and  $-I_{n-1,i} \in K_i^*$  (by (3.4)), the next to last term is nonnegative.

Similarly

$$\begin{aligned}
 \|g_{n-1,r} - g^*\|^2 &\geq \|g_{n,1} - g^*\|^2 + \|I_{n-1,1} - I_{n,1}\|^2 \\
 (4.4) \qquad &\quad - 2(g^* + b_1, I_{n-1,1} - I_{n,1}).
 \end{aligned}$$

Repeated application of (4.3) and (4.4) together with addition and the telescoping property of the last term yields

$$\begin{aligned}
 \|g - g^*\|^2 &\geq \|g_{n,r} - g^*\|^2 + \sum_{k=1}^n \sum_{\ell=1}^r \|I_{k-1,\ell} - I_{k,\ell}\|^2 \\
 (4.5) \qquad &\quad + 2 \sum_{i=1}^r (g^* + b_i, I_{n,i}) \text{ for all } n.
 \end{aligned}$$

Since the last sum is nonnegative ( $g^* + b_i \in K_i$ ,  $-I_{n,i} \in K_i^*$ ), we know

$$(4.6) \quad \sum_{k=1}^{\infty} \sum_{\ell=1}^r \|I_{k-1,\ell} - I_{k,\ell}\|^2 < \infty,$$

and hence

$$\begin{aligned}
 \|I_{n-1,\ell} - I_{n,\ell}\| &= \|g_{n,\ell-1} - g_{n,\ell}\| \quad (\ell \geq 2) \text{ and} \\
 (4.7) \qquad &\quad \|I_{n-1,1} - I_{n,1}\| = \|g_{n-1,r} - g_{n,1}\| \rightarrow 0 \text{ as } n \rightarrow \infty.
 \end{aligned}$$

We note that (4.5) implies that  $g_{n,r}$  and  $g - g_{n,r} = I_{n,1} + \dots + I_{n,r}$  are uniformly bounded.

Next we show, that there exists a subsequence  $\{n_j\}$  such that

$$(4.8) \quad \limsup_j (I_{n_j,1} + I_{n_j,2} + \dots + I_{n_j,r}, g_{n_j,1} - f) \leq 0 \quad \forall f \in \bigcap_{i=1}^r (K_i - b_i)$$

To see this, note that

$$(4.9) \quad \begin{aligned} & (I_{n,1} + \dots + I_{n,r}, g_{n,1} - f) \\ &= \sum_{i=1}^r (I_{n,i}, g_{n,1} - f - (g_{n,i} + b_i)) \quad (\text{since } (I_{n,i}, g_{n,i} + b_i) = 0) \\ &= \sum_{i=2}^r (I_{n,i}, g_{n,1} - g_{n,i}) + \sum_{i=1}^r (-I_{n,i}, f + b_i). \end{aligned}$$

The last sum is nonpositive since  $f + b_i \in K_i$  and  $-I_{n,i} \in K_i^*$ .

For the first part, we use the Cauchy-Schwarz Inequality to say

$$\begin{aligned} & \left| \sum_{i=2}^r (I_{n,i}, g_{n,1} - g_{n,i}) \right| \\ & \leq \sum_{i=2}^r \|I_{n,i}\| \|g_{n,1} - g_{n,i}\| \\ & \leq \sum_{i=2}^r \left( \sum_{m=1}^n \|I_{m,i} - I_{m-1,i}\| \right) \left( \sum_{\ell=2}^r \|g_{n,\ell-1} - g_{n,\ell}\| \right) \\ & = \left( \sum_{m=1}^n \sum_{i=2}^r \|I_{m,i} - I_{m-1,i}\| \right) \left( \sum_{\ell=2}^r \|g_{n,\ell-1} - g_{n,\ell}\| \right) \\ & = \sum_{m=1}^n a_m a_n \end{aligned}$$

where  $a_n = \sum_{\ell=2}^r \|g_{n,\ell-1} - g_{n,\ell}\| = \sum_{\ell=2}^r \|I_{n,\ell} - I_{n-1,\ell}\|$  (see (4.7)).

Since

$$a_n^2 \leq 2^{r-2} \sum_{\ell=2}^r \|I_{n,\ell} - I_{n-1,\ell}\|^2,$$

(4.6) implies that  $\sum_{n=1}^{\infty} a_n^2 < \infty$ . Thus lemma (4.1) can be employed to yield (4.8). Moreover, since the  $g_{n,r}$  are uniformly bounded, we may assume that we have chosen a subsequence such that (4.8) holds and

$g_{n_j,r}$  converges, say to  $h$ . Note that (4.7) insures that  $g_{n_j,\ell}$  also converges to  $h$  for every  $\ell$ , and hence that

$h \in \bigcap_{i=1}^r (K_i - b_i)$  (the  $K_i$  are closed). In addition, since the

$g_{n_j,r} - g = I_{n_j,1} + \dots + I_{n_j,r}$  are uniformly bounded and converge to  $h-g$ , we may use (2.4) to argue  $h = g^*$ .

Finally, in a manner similar to (4.5), we can show

$$\begin{aligned} \|g_{n_j,r} - g^*\|^2 &= \|g_{n,r} - g^*\|^2 + \sum_{m=j+1}^n \sum_{\ell=1}^r \|I_{m,\ell} - I_{m-1,\ell}\|^2 \\ (4.10) \quad &+ 2 \sum_{m=n_j+1}^n \sum_{\ell=1}^r ((g_{m,\ell} + b_\ell) - (g^* + b_\ell), I_{m-1,\ell} - I_{m,\ell}). \end{aligned}$$

We may write the last sum as

$$(4.11) \quad 2 \sum_{m=n_j+1}^n \sum_{\ell=1}^r (g_{m,\ell} + b_\ell, I_{m-1,\ell})$$

$$(4.12) \quad -2 \sum_{m=n_j+1}^n \sum_{\ell=1}^r (g_{m,\ell} + b_\ell, I_{m,\ell})$$

$$(4.13) \quad + 2 \sum_{\ell=1}^r (g^* + b_\ell, I_{n,\ell})$$

$$(4.14) \quad - 2 \sum_{\ell=1}^r (g^* + b_\ell, I_{n_j,\ell}).$$

Each term of (4.11) is nonnegative ( $g_{m,\ell} + b_\ell \in K_\ell$ ,  $-I_{m-1,\ell} \in K_\ell^*$ ).

Each term of (4.12) is zero by (2.5). Each term of (4.13) is nonnegative for the same reason as (4.11).

Finally, we can write (4.14) as

$$\begin{aligned} & -2[(g^*, \sum_{\ell=1}^r I_{n_j,\ell}) + \sum_{\ell=1}^r (b_\ell, I_{n_j,\ell})] \\ &= -2[(\lim_{j \rightarrow \infty} g_{n_j,1}, \sum_{\ell=1}^r I_{n_j,\ell}) + \sum_{\ell=1}^r (b_\ell, I_{n_j,\ell})] \\ &\rightarrow -2[\lim_{j \rightarrow \infty} [(g_{n_j,1}, \sum_{\ell=1}^r I_{n_j,\ell}) + \sum_{\ell=1}^r (b_\ell, I_{n_j,\ell})]] \\ &= -2 \lim_{j \rightarrow \infty} [\sum_{\ell=2}^r (I_{n_j,\ell}, g_{n_j,1} - g_{n_j,\ell})] \end{aligned}$$

by taking  $f$  to be  $Q$  in 4.9. This clearly equals zero by the way  $\{n_j\}_1^\infty$  was chosen. Then noting that the left side of (4.10) goes to zero as  $j \rightarrow \infty$ , it easily follows that  $g_{n,r} \rightarrow g^*$  as  $n \rightarrow \infty$ . This clearly is good enough by (4.7).

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